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MEASURES ASSOCIATED WITH TOEPLITZ MATRICES GENERATED BY THE LAURENT EXPANSION OF RATIONAL FUNCTIONS

BY

K. MICHAEL DAY

ABSTRACT. Let $T_n(a) = (a_{i-j})_{i,j=0}^n$ be the finite Toeplitz matrices generated by the Laurent expansion of an arbitrary rational function, and let $\sigma_n = \{\lambda_{n0}, \ldots, \lambda_{nn}\}$ be the corresponding sets of eigenvalues of $T_n(f)$. Define a sequence of measures α_n, $\alpha_n(E) = (n+1)^{-1}\Sigma_{\lambda_{ni}\in E} 1$, $\lambda_{ni} \in \sigma_n$, and E a set in the λ-plane. It is shown that the weak limit α of the measures α_n is unique and possesses at most two atoms, and the function f which give rise to atoms are identified.

0. Introduction. Let $f(z) = \Sigma\, a_m z^m$ be the Laurent expansion of an arbitrary rational function. Define matrices

$$T_n(f) = (a_{i-j}), \quad i, j = 0, 1, \ldots, n,$$

called Toeplitz matrices which are said to be generated by the function f. Denote by σ_n the set of $n+1$ eigenvalues of $T_n(f)$, $\sigma_n = \{\lambda_{n0}, \lambda_{n1}, \ldots, \lambda_{nn}\}$, and let $D^n(f-\lambda) = \det(T_n(f-\lambda))$. Define the set B to the the set $B = \{\lambda: \lambda = \lim_m \lambda_m, \lambda_m \in \sigma_{i_m}\}$, where $i_1, i_2, \ldots$ is an increasing subsequence of the integers. In a previous paper [2] an identity for $D^n(f-\lambda)$ was derived by means of which using the techniques of [5] it may be shown that B either is a point or consists of a finite number of nondegenerate analytic arcs.

Related to the above is the following. Define a sequence of measures α_n,

$$\alpha_n(E) = (n+1)^{-1} \sum_{\lambda_{ni}\in E} 1,$$

where $\lambda_{ni} \in \sigma_n$, and E is an arbitrary set in the λ-plane. In this paper we investigate the nature of the measures which are the weak limits of the measures α_n. It will be shown that the limit measure α is unique and possesses at most two atoms whose weight may be determined. The functions which give rise to atoms will be

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identified. For all points in the support of α which are not atoms the results of [4] show that the support of α is equal to the set B. These results, along with those appearing in [2], complete for f a rational generating function, the results of which are obtained by [5] and [4] when f is a Laurent polynomial, $f(z) = \Sigma_{-k}^{h} a_n z^n$, $h, k \geqslant 1$.

1. **Definitions.** We assume that

$$f(z) = G_s(z)/D_k(z)F_h(z) \tag{1.1}$$

where $G_s(z)$, $D_k(z)$, and $F_h(z)$ are polynomials with no common factors of exact degree s, k, and h respectively. We follow the convention that

$$D_k(z) = \prod_{i=1}^{k} (z - \delta_i), \qquad F_h(z) = \prod_{i=1}^{h} (1 - \rho_i^{-1} z),$$

the roots of $D_k(z)$ lying in the set $|z| \leqslant R_1$, and those of $F_h(z)$ in the set $|z| \geqslant R_2 > R_1$. We assume that $k \geqslant 1$, and if $h = 0$ that $s \geqslant k + 1$, so that if $s \leqslant k$ then $h \geqslant 1$ but otherwise is arbitrary. It follows that

$$\begin{aligned} f(z) - \lambda &= (G_s(z) - \lambda D_k(z)F_h(z))/D_k(z)F_h(z) \\ &= G_{k+m}(\lambda, z)/D_k(z)F_h(z) \end{aligned} \tag{1.2}$$

where $G_{k+m}(\lambda, Z) = G_s(z) - \lambda D_k(z)F_h(z)$ and $k + m = \max(s, k + h)$.

Let $f(z) = \Sigma\, a_m z^m$ be the Laurent expansion of $f(z)$ in the annulus

$$A = \{z\colon R_1 < |z| < R_2,\ 0 \leqslant R_1 < R_2 \leqslant \infty\}.$$

The conditions we assume with respect to the degrees of the polynomials $G_s(z)$, $D_k(z)$, and $F_h(z)$ result in a Laurent series representation which has nonvanishing coefficients for both positive and negative powers of z so that the matrices $T_n(f)$ are not triangular.

Let

$$Q = \{\lambda\colon G_{k+m}(\lambda, z) = 0 \text{ has multiple roots, less than } k + m \text{ roots, or } z = 0 \text{ as a root}\}. \tag{1.3}$$

From the theory of algebraic functions [3, pp. 103–104] Q is a finite set. For $\lambda \notin Q$ we may write

$$\begin{aligned} f(z) - \lambda &= C_0(\lambda) \prod_{i=1}^{k+m} (z - r_i(\lambda))/D_k(z)F_h(z), \\ C_0(\lambda) &= a,\ a - \lambda b, \text{ or } -\lambda b \end{aligned} \tag{1.4}$$

according to whether $s > k + h$, $= k + h$, or $< k + h$. Note, b is the coefficient of z^{k+h} of $D_k(z)F_h(z)$, and a is the coefficient of z^s of $G_s(z)$. If $s \leqslant k + h$ there is one value of λ for which $G_{k+m}(\lambda, z)$ has less than $k + m$ roots. Excluding the single possible value of λ for which the degree of $G_{k+m}(\lambda, z)$ is less than $k + m$, we will always assume that for fixed λ the roots $r_i(\lambda)$ of $G_{k+m}(\lambda, z) = 0$ are indexed by increasing modulus so that

$$|r_1(\lambda)| \leqslant |r_2(\lambda)| \leqslant \ldots \leqslant |r_{k+m}(\lambda)|.$$

Define the set $C = \{\lambda\colon |r_k(\lambda)| = |r_{k+1}(\lambda)|\}$. The principal result of [2] is that the set C consists of a finite number of nondegenerate analytic arcs and that the sets B and C are equal.

2. **Uniqueness of the limit measure α.** We will make use of the following theorem from potential theory [1].

THEOREM 2.1. *If α_1 and α_2 are two measures in the λ-plane such that*

$$\int \log|\lambda - \tau| d\alpha_1(\tau) = \int \log|\lambda - \tau| d\alpha_2(\tau)$$

everywhere except for a set of λ's of two-dimensional Lebesgue measure zero, then $\alpha_1 = \alpha_2$.

By the definition of the set B and the equality of the sets B and C, if N is any neighborhood of C, then the support of the measures α_n is contained in N provided n is sufficiently large. We may therefore easily prove the following proposition.

PROPOSITION 2.1. *If α is any weak limit of the measures α_n, then the support of α is contained in C.*

PROOF. If for some weak limit α the support of α is not contained in C, there would exist a continuous function f of compact support having empty intersection with C such that $\int f d\alpha \neq 0$. But there exists a neighborhood N of C having empty intersection with the support of f. Consequently for n sufficiently large, $\int f d\alpha_n = 0$. So $\lim_n \int f d\alpha_n = 0 \neq \int f d\alpha$, which is a contradiction.

We know from results in [2] that the set C consists of a finite number of analytic arcs and so is a set of two-dimensional Lebesgue measure zero. Let $f(z) - \lambda$ be as in (1.2). If $\lambda \notin C$ and $\lambda \notin Q$ (1.3), the fundamental identity of [2] states that

$$D^n(f - \lambda) = [(-1)^m C_0(\lambda)]^{n+1} \sum_I \prod_{\substack{i \in I, s \in K;\\ j \in \bar{I}, t \in H}} r_i(\lambda)^{n+1} \frac{(r_i(\lambda) - \delta_s)(\rho_t - r_j(\lambda))}{(r_i(\lambda) - r_j(\lambda))(\rho_t - \delta_s)}$$

where I runs over all subsets of order m of the set $\{1, 2, \ldots, k + m\}$, $\bar{I} = \{1, 2, \ldots, k + m\} - I$, and $K = \{1, 2, \ldots, k\}$, $H = \{1, 2, \ldots, h\}$. By the convention on the indexing of the roots of $G_{k+m}(\lambda, z)$, the term appearing in this identity which has the product

$$C_0(\lambda)^{n+1} \prod_{i=k+1}^{k+m} r_i(\lambda)^{n+1}$$

is greater in modulus than all other terms in the identity, except possibly for a finite number of integers n. Therefore

$$\lim_n D^{n+1}(f-\lambda)/D^n(f-\lambda) = C_0(\lambda) \prod_{i=k+1}^{k+m} r_i(\lambda) \neq 0. \tag{2.1}$$

Consequently if we set

$$G(\lambda) = C_0(\lambda) \prod_{i=k+1}^{k+m} r_i(\lambda), \tag{2.2}$$

then $G(\lambda)$ is an analytic function of λ for $\lambda \notin C$ as the points $\lambda \in Q$ not in C are easily seen to be removable singularities. It follows that

$$\lim_n |D^n(f-\lambda)|^{1/n+1} = |G(\lambda)| \neq 0 \tag{2.3}$$

except on a set of two-dimensional Lebesgue measure zero. From (2.3) we obtain

$$\lim_n \log |D^n(f-\lambda)|^{1/n+1} = \log|G(\lambda)|. \tag{2.4}$$

But the left-hand side of (2.4) can be written

$$\begin{aligned} \lim_n 1/n+1 \cdot \log|D^n(f-\lambda)| &= \lim_n 1/n+1 \cdot \sum_{i=0}^{n} \log|\lambda - \lambda_{ni}| \\ &= \lim_n \int \log|\lambda - \tau| d\alpha_n(\tau). \end{aligned} \tag{2.5}$$

From (2.4) and (2.5) then

$$\lim_n \int \log|\lambda - \tau| d\alpha_n(\tau) = \log|G(\lambda)|. \tag{2.6}$$

THEOREM 2.2. *The measures α_n converge weakly to a measure α with support contained in C.*

PROOF. If α is any weak limit of the measures α_n, by (2.6),

$$\int \log|\lambda - \tau| d\alpha(\tau) = \log|G(\lambda)|$$

except for a set of two-dimensional Lebesgue measure zero. Consequently by Theorem 2.1 the measure α is unique.

3. **The structure of the limit measure α.** In this section we show that the limit measure α can have at most two atoms.

THEOREM 3.1. *Let $f(z)$ satisfy the conditions of* §1. *Then* $f(z) - \lambda = G_{k+m}(\lambda, z)/D_k(z)F_h(z)$ *as given by* (1.2). *If the minimum degree of* $G_{k+m}(\lambda, z)$ *for all* λ *as a polynomial in* z *is greater than or equal to* k, *then* α *has at most one atom. If the minimum degree of* $G_{k+m}(\lambda, z)$ *is less than* k, *then* α *has at least one atom but at most two atoms. If for* $\lambda = \lambda_1$,

$$G_{k+m}(\lambda_1, z) = z^{k+d'}G_{s'-k-d'}(z)$$

with $d' \geqslant 1$, $s' \geqslant k + d'$, *and* $G_{s'-k-d'}(0) \neq 0$, *then* λ_1 *is an atom of weight* $d'/k + d'$. *If for* $\lambda = \lambda_2$ *and* $s'' < k$, $G_{k+m}(\lambda_2, z) = G_{s''}(z)$ *with* $s'' + d'' = k$, $d'' \geqslant 1$, *then necessarily* $h \geqslant 1$ *and* λ_2 *is an atom of weight* $d''/h + d''$. *Conversely if the limit measure* α *has an atom at* $\lambda = \lambda_0$, *then* $G_{k+m}(\lambda_0, z)$ *must assume one of the two forms given above. Each type of atom can occur at most once, though the same function may have associated with it an atom of both types. When this occurs the sum of the weights of the atoms is less than one.*

PROOF. ($\longrightarrow$) Without loss of generality we may assume that $\lambda_1 = 0$. Then $f(z)$ has the form

$$f(z) = z^{k+d'}G_{s-k-d'}(z)/D_k(z)F_h(z). \tag{3.1}$$

To prove that $\lambda = 0$ is an atom of weight $d'/k + d'$ we will prove that, in a neighborhood of $\lambda = 0$, $D^n(f - \lambda) = \lambda^{\tau_n} \cdot O(1)$ where τ_n is an integer such that $\lim_n \tau_n/n + 1 = d'/k + d'$. It is clear from (1.1) and (1.4) that

$$\lim_{\lambda\to 0} C_0(\lambda)\prod_{i=1}^{k+m}(z - r_i(\lambda)) = z^{k+d'}G_{s-k-d'}(z),$$

and therefore in a neighborhood of $\lambda = 0$ that

$$C_0(\lambda)\prod_{i=k+d'+1}^{k+m} r_i(\lambda) = O(1), \tag{3.2}$$

that is, is bounded away from zero and infinity.

Results from the theory of algebraic functions tell us [3, pp. 99–103] that from small $|\lambda|$

$$r_i(\lambda) = \sum_{j=1}^{\infty} C_j\lambda^{j/k+d'}, \qquad i = 1, \ldots, k + d', \tag{3.3}$$

each $r_i(\lambda)$ being determined by each determination of the root $\lambda^{1/k+d'}$, and for $k + h > s$

$$r_i(\lambda) = \lambda^{-1/k+h-s}\sum_{j=0}^{\infty} b_j^{j/k+h-s}, \qquad i = s + 1, \ldots, k + h, \tag{3.4}$$

each $r_i(\lambda)$ being determined by each determination of the root $\lambda^{1/k+h-s}$.

Referring to the identity for $D^n(f-\lambda)$, for $I_0 = \{k+1, \ldots, k+m\}$ there are d' roots near $z = 0$ in the product $\Pi_{i\in I_0} r_i(\lambda)$. By (3.2),(3.3), and (3.4),

$$C_0(\lambda)^{n+1} \prod_{i\in I_0} r_i(\lambda)^{n+1} = \lambda^{d'(n+1)/(k+d')} \cdot O(1). \tag{3.5}$$

For all other indices $I \neq I_0$, there are at least $d' + 1$ roots $r_i(\lambda)$ near zero appearing in the factor $\Pi_{i\in I} r_i(\lambda)$. Therefore for all $I \neq I_0$,

$$C_0(\lambda)^{n+1} \prod_{i\in I} r_i(\lambda)^{n+1} = \lambda^{d'(n+1)/(k+d')} \cdot o(1). \tag{3.6}$$

Each of the factors

$$\prod_{\substack{i\in I, s\in K;\\ j\in \bar{I}, t\in H}} \frac{(r_i-\delta_s)(\rho_t-r_j)}{(r_i-r_j)(\rho_t-\delta_s)} \tag{3.7}$$

is of fixed order independent of n. By (3.3) and (3.4) these factors are for small $|\lambda|$ of order greater than $O(|\lambda|^{k+m})$ and smaller than $O(|\lambda|^{-(k+m)})$. Consequently by (3.5) and (3.6) $D^n(f-\lambda) = \lambda^{\tau_n} \cdot O(1)$ where τ_n is an integer satisfying $d'(n+1)/(k+d') - (k+m) \leqslant \tau_n \leqslant d'(n+1)/(k+d') + (k+m)$. Clearly $\lim_n \tau_n/(n+1) = d'/(k+d')$ so that $\lambda = 0$ is an atom of weight $d'/(k+d')$. If there were another value $\lambda = a \neq 0$, where $G_{k+m}(a, z) = z^{k+d^*} G_{s'-k-d^*}(z)$ with $d^* \geqslant 1$, and $G_{s'-k-d^*}(0) \neq 0$, then

$$f(z) = z^{k+d^*} G_{s'-k-d^*}(z)/D_k(z)F_h(z) + a$$

which contradicts (3.1).

Assume now for $\lambda_2 = 0$ that

$$f(z) = G_{s''}(z)/D_k(z)F_h(z) \tag{3.8}$$

with $s'' + d'' = k$, $d'' \geqslant 1$. Since $f(z)$ satisfies conditions given in §1, necessarily $h \geqslant 1$, and $k + m = k + h$. By (3.4) the roots $r_i(\lambda)$ have the form

$$r_i(\lambda) = \lambda^{-1/h+d''} \cdot O(1), \qquad i = s+1, \ldots, k+h.$$

In this instance by (1.4) $C_0(\lambda) = -\lambda b$. Consequently for $I_0 = \{k+1, \ldots, k+h\}$,

$$C_0(\lambda) \cdot \prod_{i\in I_0} r_i(\lambda) = \lambda^{d''/h+d''} \cdot O(1),$$

and for all $I \neq I_0$,

$$C_0(\lambda) \cdot \prod_{i\in I} r_i(\lambda) = \lambda^{d''/h+d''} \cdot o(1),$$

and the argument employed previously shows that $D^n(f-\lambda) = \lambda^{\tau_n} \cdot O(1)$ where τ_n is an integer such that $\lim_n \tau_n/(n+1) = d''/(h+d'')$. For all $\lambda \neq 0$ clearly the degree of $G_{k+h}(\lambda, z)$ is $k+h$ and so an atom of this type can occur at most once.

If there is an atom of each type then $k+m = k+h$ with $h \geqslant 1$. To maximize $d'/(k+d')$ requires that $d' = h$. To maximize $d''/(h+d'')$ requires that $d'' = k$. We would then have that

$$(d'/(k+d')) + (d''/(h+d'')) = (h/(k+h)) + (k/(h+k)) = 1,$$

but $f(z)$ would have the form $f(z) = (az^{k+h} + b)/(cz^{k+h} + d)$ with $ac \neq 0, ab - cd \neq 0$. But this is impossible since necessarily with $h, k \geqslant 1$ the denominator must have two zeros of different modulus.

($\longleftarrow$) Assume that $\lambda = 0$ is an atom of the limit measure α. For $\lambda \notin C$ and Q of (1.3) we have seen by (2.2) and (2.6) that

$$\lim_n \int \log|\lambda - \tau| d\alpha_n(\tau) = \log\left|C_0(\lambda) \prod_{i=k+1}^{k+m} r_i(\lambda)\right|. \tag{3.9}$$

We may rewrite the left-hand side of (3.9) so that

$$\int_{/\{0\}} \log|\lambda - \tau| d\alpha(\tau) + \rho \log|\lambda| = \log\left|C_0(\lambda) \prod_{i=k+1}^{k+m} r_i(\lambda)\right|$$

where ρ is the weight of α at $\lambda = 0$.

Since $\log|\lambda| \longrightarrow -\infty$ as λ tends to zero, it follows that $\log|C_0(\lambda)\Pi_{i=k+1}^{k+m} r_i(\lambda)| \longrightarrow -\infty$ also. We assume as always that the indices are arranged by increasing moduli. If for $f(z) = G_s(z)/D_k(z)F_h(z)$ we assume $s \geqslant k+h$, then the polynomial $G_{k+m}(\lambda, z)$ has degree s. By (1.4) $C_0(\lambda)$ Is bounded away from zero, and the roots $r_i(\lambda)$, $i = 1, \ldots, s$, are bounded for small $|\lambda|$. Consequently $\log|\Pi_{i=k+1}^s r_i(\lambda)| \longrightarrow -\infty$, and so $|\Pi_{i=k+1}^s r_i(\lambda)| \longrightarrow 0$ as λ tends to zero. Let d be the smallest integer such that $\Pi_{i=k+d+1}^s r_i(\lambda)$ is bounded away from zero as $\lambda \longrightarrow 0$. Necessarily $1 \leqslant d \leqslant s-k-1$. By the convention regarding indexing $r_i(\lambda)$, $i = k + d$, and the roots $r_1(\lambda)$, $i = k+d+1, \ldots, s$, are bounded away from zero. We may conclude that

$$f(z) = z^{k+d} G_{s-k-d}(z)/D_k(z)F_h(z)$$

with $d \geqslant 1$, and $G_{s-k-d}(0) \neq 0$.

If we assume $s < k+h$ so that $k+m = k+h$, then $h \geqslant 1$ and by (1.4) $C_0(\lambda) = -\lambda b$. If $k+1 \leqslant s < k+h$ then, by (3.4), $C_0(\lambda)\Pi_{s+1}^{k+h} r_i(\lambda) = O(1)$ as $\lambda \longrightarrow 0$. But then $\log|\Pi_{k+1}^s r_i(\lambda)| \longrightarrow -\infty$, and so $|\Pi_{i=k+1}^s r_i(\lambda)| \longrightarrow 0$ as λ tends to zero. As before let d be the smallest integer such that $\Pi_{i=k+d+1}^s r_i(\lambda)$ is bounded away from zero as λ tends to zero, $1 \leqslant d \leqslant s-k-1$. But then $r_i(\lambda) \to 0$, $i = 1, \ldots, k+d$, and we may again conclude that

$$f(z) = z^{k+d} G_{s-k-d}(z)/D_k(z)F_h(z)$$

with $d \geqslant 1$, and $G_{s-k-d}(0) \neq 0$.

If $s = k$ then again, by (3.4), $C_0(\lambda)\Pi_{k+1}^{k+h} r_i(\lambda) = O(1)$ for small $|\lambda|$, but this contradicts the fact that $\log |C_0(\lambda)\Pi_{k+1}^{k+h} r_i(\lambda)| \to -\infty$. The only possibility that remains is that $f(z) = G_s(z)/D_k(z)F_h(z)$ with $s < k$. But then $\lambda = 0$ is an atom of the second type. This completes the proof of the theorem.

An example of a function with two atoms is the following. Let

$$f(z) = z^4/(z^4 - z - a) = z^4/D_2(z)F_2(z)$$

where a is any number such that the moduli of the roots of the denominator are distinct. Then $\lambda = 0$ is an atom of weight $2/(2+2) = 1/2$. Since $f(z) - 1 = (z+a)/D_2(z)F_2(z)$, $\lambda = 1$ is an atom of weight $1/(2+1) = 1/3$. More generally the reader may verify that there exists a rational function whose limit measure α has atoms at two arbitrary points in the λ-plane of arbitrary fractional weight as long as the sum of these weights is less than one.

I. I. Hirschman in his paper [4] derives an explicit formula for the limit measure α when the generating function f is a Laurent polynomial. To do so he makes use of the result that the support of any limit measure α is contained in the set C which consists of a finite number of nondegenerate analytic arcs. He proves that for $\lambda \notin C$ that

$$\lim_n D^{n+1}(f-\lambda)/D^n(f-\lambda) = G(\lambda) \neq 0$$

exists. By means of this function he derives a formula which shows that the limit measure is unique, has support of all of C, and is absolutely continuous. His proof may be applied to the case we are considering except in one situation. His analysis requires that there do not exist any atoms in the limit measure. To do this he must show that $G(\lambda) \not\to 0$ as λ tends to any point in C.

We also have seen that for $\lambda \notin C$ that

$$\lim D^{n+1}(f-\lambda)/D^n(f-\lambda) = G(\lambda) \neq 0$$

exists, but we cannot show that $G(\lambda) \not\to 0$ as λ tends to an arbitrary point of C. But the analysis shows that if this occurs then any such point must be an atom. We may conclude that the limit measure α has support C and is absolutely continuous except for the possible existence of two atoms. If the limit measure has an atom its weight may be determined by Theorem 3.1, and the formula derived by Hirschman gives the measure at all other points of C where α is absolutely continuous. J. L. Ullman has shown for the Laurent polynomial case that C is connected [6]. His proof may be applied here. The significance of connectedness in the case that α is apparent. It is not known whether or not the set C may separate the plane.

REFERENCES

1. L. Carleson, *Mergelyan's theorem on uniform polynomial approximation*, Math. Scand. **15** (1964), 167–175. MR 33 #6368.

2. K. M. Day, *Toeplitz matrices generated by the Laurent series expansion of an arbitrary rational function*, Trans. Amer. Math. Soc. **206** (1975), 224–245.

3. E. Hille, *Analytic function theory*. Vol. II, Introduction to Higher Math., Ginn, Boston, Mass., 1962. MR 34 #1490.

4. I. I. Hirschman, Jr., *The spectra of certain Toeplitz matrices*, Illinois J. Math. **11** (1967), 145–159. MR 34 #4905.

5. P. Schmidt and F. Spitzer, *The Toeplitz matrices of an arbitrary Laurent polynomial*, Math. Scand. **8** (1960), 15–38. MR 23 #A1977.

6. J. L. Ullman, *A problem of Schmidt and Spitzer*, Bull. Amer. Math. Soc. **73** (1967), 883–885. MR 36 #3056.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MICHIGAN 48104